Further results on switched control of linear systems with constraints

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Further results on switched control of linear systems with constraints[1]

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Abstract

In a previous paper ([2], see also [4]) we proposed a supervisory control system to *globally* regulate to zero the state of a very poorly modeled, open-loop unstable but not exponentially unstable, linear process in the presence of input constraints. The process to control was unknown but assumed to belong to a *finite* family $\mathcal{P}$ of nominal models. In this paper, the analysis is extended to the case in which $\mathcal{P}$ is *not* finite and is a closed, bounded subset of a real, finite-dimensional, normed linear space. In this analysis, the property of the multi-estimator/multi-controller of being "robustly" integral input-to state stable is exploited.

1 Introduction

Logic-based switched controllers provide a systematic approach to the problem of controlling processes whose model is very uncertain. By very uncertain model, it is typically meant that the *actual* model of the process $\mathbb{P}$ is an unknown member of a family of systems of the form $\mathcal{F} = \cup_{p\in\mathcal{P}}\{\mathbb{N}_p\}$ where each $\mathbb{N}_p$ is a given *nominal process model* and $\mathcal{P}$ is an *infinite* set of indices or points. The approach relies on a family of candidate controllers $\mathcal{C} := \{\mathbb{C}_p : p \in \mathcal{P}\}$ and on a supervisor – generating a piece-wise constant signal σ which takes on values in $\mathcal{P}$. The candidate controller $\mathbb{C}_p$ is designed to control the nominal process model $\mathbb{N}_p$. The role of the supervisor is to choose within the family $\mathcal{C}$ from time to time the controller to be put in the feedback loop, thus realizing the switched controller $\mathbb{C}_\sigma$. An estimation-based supervisor consists of three subsystems, a multi-estimator $\mathbb{E}$, a bank of monitoring signal generators $\mathbb{M}_p$, $p \in \mathcal{P}$, and a switching logic $\mathbb{S}$. $\mathbb{E}$ is a system with state x whose input is the pair of input and output vectors of $\mathbb{P}$, and whose p-th output is a signal y_p. The multi-estimator is designed in such a way that the behavior from its input to y_p coincides with the input/output behavior of the process provided that $\mathbb{N}_p$ is the actual process model and no measurement noise or disturbances are present. A *monitoring signal generator* $\mathbb{M}_p$ is a system whose input is the p-th output estimation error $e_p := y_p - y$ and whose output μ_p is a suitably defined signal which measures the size of the e_p. The third subsystem of an estimator-based supervisor is a switching logic $\mathbb{S}$ whose role is to generate σ by assessing the signals μ_p's.

The estimation-based supervisory control for *linear* systems is now very well-understood ([14, 13]). By exploiting the exponential stability property, it is possible to devise a logic that allowing sufficiently slow switching among the candidate controllers guarantees the desired state regulation of the unknown process. The interest in the present paper, as well as in the papers [2, 4], is focused on the same problem of regulation for linear uncertain processes, but with the additional presence of *constraints* on the input. The constraints make the problem very difficult because the usual tools which have been used to attack the problem cannot be utilized anymore. Indeed, no control law exists which allow to achieve global exponential stability when the input is constrained. Neither other properties such as input-to-state stability (ISS) ([16]) which have been successfully used in [9] to deal with supervisory control of nonlinear systems can be guaranteed unless an unpractical restriction on the magnitude of the output estimation errors e_p's is assumed. Nevertheless, it has been shown in [2, 4] that, relying on a weaker property than ISS, i.e. integral ISS or iISS ([17]), and on a new switching logic which *adjusts* at each switching instant the time needed by each controller to be in the loop before being replaced by a different controller, it is possible to design a supervisory control architecture capable of stabilizing the state of the process despite of the uncertainty and the input constraint. An important feature of the supervisor is that it is guaranteed to well-perform even in the case the switching never stops, as it is often the case in practical situations.

In [2, 4], the interest was centered around the case in which the possible nominal models for the unknown process belong to a family with a *finite* number of elements. The present contribution shows how to extend the results of [2, 4] to the case in which the parameter p representing the uncertainty in the process does not take on a finite number of values but rather ranges in a continuum of points. The main technical

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concept the result of this paper is based on is that of iISS which is robust to arbitrarily small parameter mismatch (cf. Section 4). For the class of systems we are interested in, such a robust iISS can be established by following the arguments in [12, 18, 3]. Among the systems for which the results of the paper are applicable, we point out for instance the chain of integrators with *unknown sign*, i.e.

$$\dot{x}_1 = p_1 x_2 \ , \ \dot{x}_2 = p_2 x_3 \ , \ \ldots \ , \dot{x}_n = p_n \mathrm{sat}(v) \ ,$$

where the p_i's satisfy $0 < \underline{p} \leq |p_i| \leq \bar{p}$ but are otherwise unknown.

The main result of the paper is Theorem 1 in Section 6. Before that, the formalization of the problem and the class of systems of interest are given in Section 2. Sections 3, 4, 5 contain the description of the components the supervisory control architecture is composed of.

2 Problem Formulation

Consider a process $\mathbb{P}$ which is unknown but is assumed to admit the model of a SISO linear system, whose transfer function is a member of the known class

$$\mathcal{C}_{\mathbb{P}} = \bigcup_{p \in \mathcal{P}} \{\nu_p(s)\} \ , \tag{1}$$

where $\mathcal{P}$ is a closed, bounded subset of a real, finite-dimensional, normed linear space and

$$\nu_p(s) = \frac{\alpha_p(s)}{\beta_p(s)}$$

is a strictly proper transfer function with $\beta_p(s)$ a monic polynomial and $\alpha_p(s)$, $\beta_p(s)$ coprime polynomials. Assume that

Assumption 1 *For any $p \in \mathcal{P}$, all the poles of $\nu_p(s)$ lie in the closed left-half plane.*

Let n_ν be an upper bound on the McMillan degree of each ν_p and $(A_{\mathrm{E}}, b_{\mathrm{E}})$, with $A_{\mathrm{E}} \in \mathbb{R}^{n_\nu \times n_\nu}$ and $b_{\mathrm{E}} \in \mathbb{R}^{n_\nu \times 1}$, a controllable pair with A_{E} Hurwitz. Then ([14], Section IV) for each $p \in \mathcal{P}$ there exists a $c_p \in \mathbb{R}^{1 \times 2n_\nu}$ such that the triple

$$(c_p, \bar{A}_p, b) := \left(c_p, \begin{bmatrix} A_{\mathrm{E}} & 0 \\ 0 & A_{\mathrm{E}} \end{bmatrix} + \begin{bmatrix} b_{\mathrm{E}} \\ 0 \end{bmatrix} c_p, \begin{bmatrix} 0 \\ b_{\mathrm{E}} \end{bmatrix} \right) \tag{2}$$

is a stabilizable and detectable realization of $\nu_p(s)$. Note that even if the models in $\mathcal{C}_{\mathbb{P}}$ may significantly differ from each other, their state space realizations

$$\mathbb{N}_p : \quad \begin{array}{rcl} \dot{x}_p & = & \bar{A}_p x_p + bu \\ y & = & c_p x_p \ , \end{array} \tag{3}$$

have all the same dimension and the dependence on the parameter p is summarized in the vector c_p only, which is required to satisfy the following:

Assumption 2 *The function $p \to c_p$ is continuous.*

The constraints on the input are taken into account by introducing the saturation function $\mathrm{sat}(\cdot)$, which has the following properties (see e.g. [10]).

Definition. A locally Lipschitz function $\mathrm{sat}(\cdot) : \mathbb{R} \to \mathbb{R}$ is said to be a saturation function if

(i) $\mathrm{sat}(0) = 0$ and $r\mathrm{sat}(r) > 0$ for all $r \neq 0$,

(ii) there exist $\underline{k}, \bar{k} > 0$ such that $|\mathrm{sat}(r)| \leq \bar{k}$ for all r and $\liminf_{|r| \to \infty} |\mathrm{sat}(r)| \geq \underline{k}$,

(iii) $\mathrm{sat}(r)$ is differentiable in a neighborhood of the origin and $\mathrm{sat}'(0) = 1$. $\triangleleft$

Incorporating the saturation function in the models (3), we obtain the nonlinear systems

$$\bar{\mathbb{N}}_p : \quad \begin{array}{rcl} \dot{x}_p & = & \bar{A}_p x_p + b\,\mathrm{sat}(v) \\ y & = & c_p x_p \end{array} \tag{4}$$

where v is generated by the switched controller $\mathbb{C}_\sigma$.

As in [2, 4], also in this paper no unmodeled dynamics affect the plant, i.e. the "exact matching" case is considered. This means that the model $\tilde{\mathbb{P}}$ of the plant with input constraints, namely

$$\begin{array}{rcl} \dot{x}_{\mathbb{P}} & = & A_{\mathbb{P}} x_{\mathbb{P}} + b\,\mathrm{sat}(v) \\ y & = & c_{\mathbb{P}} x_{\mathbb{P}} \ , \end{array} \tag{5}$$

with $x_{\mathbb{P}} \in \mathbb{R}^n$, $n = 2n_\nu$, $v \in \mathbb{R}$, $y \in \mathbb{R}$, is such that there exists a parameter p^* for which

$$A_{\mathbb{P}} = \bar{A}_{p^*} \ , \quad c_{\mathbb{P}} = c_{p^*} \ . \tag{6}$$

The feedback loop we are considering is thus that depicted in Figure 1. The control problem is to design a multi-controller $\mathbb{C}_\sigma$ and the supervisory architecture which acts on $\mathbb{C}_\sigma$ through the switching signal σ so as to globally regulate to zero the state of the process $\mathbb{P}$ despite of the large uncertainty on its model.

3 Identifier-based Multi-estimator and Monitoring Signal Generator

We consider, as in [14], a (*state-shared*) multi-estimator described by the equations

$$\begin{array}{rcl} \dot{x} & = & \begin{bmatrix} A_{\mathrm{E}} & 0 \\ 0 & A_{\mathrm{E}} \end{bmatrix} x + \begin{bmatrix} 0 \\ b_{\mathrm{E}} \end{bmatrix} \mathrm{sat}(v) + \begin{bmatrix} b_{\mathrm{E}} \\ 0 \end{bmatrix} y \\ y_p & = & c_p x \ , \quad p \in \mathcal{P} \ . \end{array} \tag{7}$$

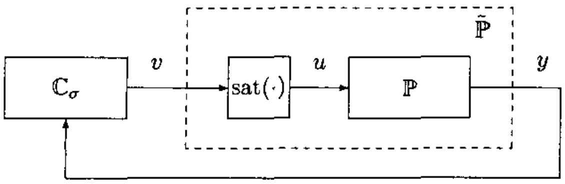

Figure 1: Feedback interconnection.

Multi-estimator (7) can also be rewritten in the more compact form

$$\begin{aligned} \dot{x} &= Ax + b\,\mathrm{sat}(v) + dy \\ y_p &= c_p x\,, \quad p \in \mathcal{P}\,. \end{aligned} \tag{8}$$

The outputs y_p, $p \in \mathcal{P}$, generated by the multi-estimator (8) are used to obtain the *output estimation errors* $e_p = y_p - y$ which feed the monitoring signal generators $\mathbb{M}_p$

$$\mathbb{M}_p: \quad \dot{\mu}_p = -\lambda\mu_p + |e_p|^2\,,\ \mu_p(0) > 0\,,\ p \in \mathcal{P}\,. \tag{9}$$

The monitoring signal generators are input-to-state stable, provided that $\lambda > 0$. Also note that the exact matching condition (6) and the equations of the output estimation errors show that e_{p^*} decays exponentially to zero, i.e. $|e_{p^*}(t)| \le \bar{C}\exp(-\bar{\lambda}t)$, for some positive numbers $\bar{C}, \bar{\lambda}$.

Of course, in view of the nature of $\mathcal{P}$, equations (9) cannot be implemented. An implementable equivalent way to generate the monitoring signal μ_p makes use of a weighting matrix W ([14]), which is generated by a (finite dimensional) causal dynamical systems whose inputs are x and y:

$$\tilde{\mathbb{M}}: \quad \dot{W} = -2\lambda W + \begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}'\,. \tag{10}$$

It is easy to verify that given W, solution of (10), the monitoring signal can be computed by the relation

$$\mu_p(t) = [c_p \quad -1]W(t)[c_p \quad -1]'\,. \tag{11}$$

4 Multi-controller

Following a standard approach in supervisory control (cf. [9, 7, 4]), the controller is designed for a system obtained from the multi-estimator (8) by "injecting" the variable $y = y_p - e_p$, thus making the system input-output equivalent to the p-th model $\mathbb{N}_p$. The resulting system – keeping in mind that $\bar{A}_p = (A + dc_p)$ – is described by equations of the form

$$\begin{aligned} \dot{x} &= \bar{A}_p x + b\,\mathrm{sat}(v) - de_p \\ y_p &= c_p x\,. \end{aligned} \tag{12}$$

It can be shown (see for instance [3, 5]) that since, for each $p \in \mathcal{P}$, the pair $(\bar{A}_p, b)$ is stabilizable and $\bar{A}_p$ has no eigenvalue in the open right-half plane of the complex plane, system (12) can be made integral input-to-state stable (iISS) and locally exponentially stable with a suitable feedback. We recall that [16]

Definition. A system $\dot{\xi} = f(\xi, u)$ is iISS if there exist functions[1] $\alpha(\cdot), \tilde{\theta}_1(\cdot), \tilde{\theta}_2(\cdot) \in \mathcal{K}_\infty$, $\gamma(\cdot) \in \mathcal{K}$, such that for all ξ_0, all u, and for all $t \ge 0$,

$$\alpha(|\xi(t, \xi_0, u)|) \le \tilde{\theta}_1(\tilde{\theta}_2(|\xi_0|)e^{-t}) + \int_0^t \gamma(|u(s)|)ds\,. \tag{13}$$

The function $\gamma(\cdot)$ is sometimes referred to as the gain function.

Lemma 1 ([3], [18]) *For each $p \in \mathcal{P}$, there exists a feedback law $v = \chi_p(x)$, such that the closed-loop system*

$$\dot{x} = \bar{A}_p x + b\,\mathrm{sat}(\chi_p(x)) - de_p\,, \tag{14}$$

is iISS with respect to e_p with quadratic gain function. In particular, there exist class-$\mathcal{K}_\infty$ functions $\alpha(\cdot), \tilde{\theta}_1(\cdot), \tilde{\theta}_2(\cdot)$, and a constant $\bar{\gamma} > 0$ such that the solution $x(t)$ of (14) from the initial condition $x(t_0) = x_0$ under the input e_p satisfies, for all $t \ge t_0 \ge 0$, all x_0 and all e_p,

$$\alpha(|x(t)|) \le \tilde{\theta}_1(\tilde{\theta}_2(|x_0|)e^{-(t-t_0)}) + \int_{t_0}^t \bar{\gamma}|e_p(\tau)|^2 d\tau\,. \tag{15}$$

Also, there exist positive real numbers $a_1, a_2, a_3, \bar{s}$, and smooth functions $W_p : \mathbb{R}^{2n_\nu} \to \mathbb{R}$, such that $a_1|x|^2 \le W_p(x) \le a_2|x|^2$ and

$$\frac{\partial W_p}{\partial x}(\bar{A}_p x + b\,\mathrm{sat}(\chi_p(x))) \le -a_3|x|^2 \tag{16}$$

for all $|x| \in [0, \bar{s}]$.

In the present setting, in which $\mathcal{P}$ consists of a continuum of points, we make use of a family of controllers $\mathcal{C} = \{\mathbb{C}_p : p \in \mathcal{P}\}$, which guarantees stronger stability properties, namely, we assume that the controller makes the system "robustly" integral input-to-state stable.

Assumption 3 *There exist an $\varepsilon > 0$ and a family of feedback laws $\{v = \overline{\chi}_p(x) : p \in \mathcal{P}\}$ such that for each $p, q \in \mathcal{P}$, with $|p - q| \le \varepsilon$, the system*

$$\dot{x} = \bar{A}_p x + b\,\mathrm{sat}(\overline{\chi}_q(x)) - de_p\,, \tag{17}$$

[1] $\mathcal{K}$ is the class of functions $[0, \infty) \to [0, \infty)$ which are zero at zero, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subclass of functions $\mathcal{K}$ which are unbounded.

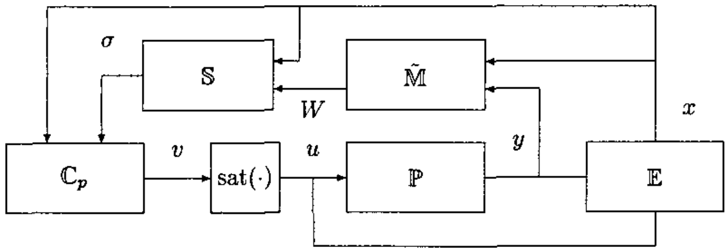

Figure 2: Supervisory control system in the presence of input saturation.

is iISS with respect to e_p with quadratic gain function. In particular, there exist class-$\mathcal{K}_\infty$ functions $\alpha(\cdot), \tilde{\theta}_1(\cdot), \tilde{\theta}_2(\cdot)$, and a constant $\bar{\gamma} > 0$ such that the solution $x(t)$ of (17) from the initial condition $x(t_0) = x_0$ under the input e_p satisfies (15), for all $t \geq t_0 \geq 0$, all x_0 and all e_p.
Also, there exist positive real numbers $a_1, a_2, a_3, \bar{s}$, and smooth functions $W_p : \mathbb{R}^{2n_\nu} \to \mathbb{R}$, such that $a_1|x|^2 \leq W_p(x) \leq a_2|x|^2$ and (16) holds for all $|x| \in [0, \bar{s}]$.

Remark. If matrix A_p in system (17) is neutrally stable, then the control law $\chi_q(x)$ is actually linear, i.e. there exists a matrix $\bar{F}_q$ for which $\chi_q(x) = \bar{F}_q x$. Therefore, system (17) with $e_p = 0$ can be rewritten as $\dot{x} = \bar{A}_p x + b\,\text{sat}(\bar{F}_p x + v)$, where $v = (F_q - F_p)x$. The results in [11] guarantee the existence of a Lyapunov function $V(x)$ and a number $\lambda > 0$ for which $\dot{V} \leq -|x|^2 + \lambda|x||v|$. Therefore, letting ε be such that $|q - p| \leq \varepsilon$ implies $\|F_q - F_p\| \leq 1/(2\lambda)$, asymptotic stability of the system is drawn. If system (17) has an $\mathcal{L}_2$-to-$\mathcal{L}_\infty$-like stability property when $e_p \neq 0$, then using the same arguments of the proof of Lemma 4 in [3], it is also possible to prove integral ISS of (17) with respect to e_p with a quadratic gain function. For more general matrices A_p's, similar conclusion can be drawn using arguments of the kind found in [18] and [12]. $\triangleleft$

5 Switching logic

The last component of the supervisory control architecture, namely switching logic $\mathbb{S}$, is described in this section. $\mathbb{S}$ is the recently introduced ([3, 5]) *adjustable* dwell-time switching logic. The switching logic is designed as a hybrid dynamical system whose inputs are x and W and whose state is composed by a discrete-time variable $X \in \mathbb{R}$, a continuous-time variable τ (*timing signal*) and the piece-wise constant signal $\sigma : [0, \infty) \to \mathcal{P}$. To describe the functioning of $\mathbb{S}$ we need to introduce some notation.
Let $\alpha(\cdot)$, $\tilde{\theta}_1(\cdot)$, $\tilde{\theta}_2(\cdot) \in \mathcal{K}_\infty$ and $a_1, a_2, a_3, \bar{s} > 0$ be as in Assumption 3. Define the functions

$$\theta_1(r) := \tilde{\theta}_1^{-1}(\alpha(r/3)/2), \quad \theta_2(r) := \tilde{\theta}_2(r), \qquad (18)$$

and set[2]

$$\tau_\Delta(r) := \ln(\theta_2(r)/\theta_1(r)), \quad r > 0. \qquad (19)$$

Let $\bar{r} := \theta_2^{-1}(\theta_1(3\bar{s}))$, and fix a "dwell-time" function $\tau_D : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ satisfying

$$\tau_D(r) \geq \begin{cases} \tau_\Delta(r), & r \geq \bar{r} \\ \max\{\tau_\Delta(\bar{r}), \dfrac{3a_2}{a_3}\ln\dfrac{a_2}{a_1}\}, & r < \bar{r}. \end{cases} \qquad (20)$$

Adjustable Dwell-Time Switching Logic $\mathbb{S}$ ([3, 5]). Set $\sigma(0) = \text{argmin}_{p\in\mathcal{P}}\{\mu_p(0)\}$. Suppose that at some time t_0, σ has just changed its value to p. At this time, the timing signal τ is reset to 0 and a variable X is set equal to $|x(t_0)|$, that is in X is "stored" the magnitude of the state of the plant at that switching time. Compute now the *dwell-time* $\tau_D(X)$. At the end of the switching period, when $\tau = \tau_D(X)$, if there exists the minimal value $q \in \mathcal{P}$ such that μ_q is smaller than μ_σ, then σ is set equal to q, τ is reset to zero and the entire process is repeated. Otherwise, a new search for the minimal value $q \in \mathcal{P}$ such that μ_q is smaller than μ_σ is carried out.

6 Main Result

We can now analyze the supervisory control system that we have introduced in the previous sections. The multi-estimator $\mathbb{E}$ described by the equations (8), the family of controllers $\mathcal{C} = \{\mathbb{C}_p : p \in \mathcal{P}\}$ described in Assumption 3, the monitoring signal generators $\tilde{\mathbb{M}}_p$ characterized by equation (10) and the switching logic $\mathbb{S}$ compose the switching controller

$$\mathbb{C}_\sigma : \quad \begin{aligned} \dot{x} &= Ax + b\,\text{sat}(\overline{\chi}_\sigma(x)) + dy \\ v &= \overline{\chi}_\sigma(x) \end{aligned} \qquad (21)$$

The closed loop system to analyze is composed by the unknown process $\tilde{\mathbb{P}}$ of the form (5) and the switching controller (21) (see Figure 2).

First of all we note the following

[2] Note that $\theta_2(r)/\theta_1(r) > 1$ for all $r > 0$, and (19) is well-posed (cf. [3, 5]).

Fact 1 *If Assumption 3 holds, then for each set of initial conditions $x_{\mathbb{P}}(0)$, $x(0)$, $\mu_p(0) > 0$, $p \in \mathcal{P}$, $\sigma(0)$, the responses of the supervisory control system (8), (21), and (9), and of the process (5) exist for all $t \in [0, \infty)$.*

Indeed, system (21) can be rewritten as

$$\dot{x} = \bar{A}_\sigma x + b\,\mathrm{sat}(\overline{\chi}_\sigma(x)) - de_\sigma \ , \tag{22}$$

and the property is an easy consequence of the integral input-to-state stability of the system with respect to e_σ, for any fixed value of σ.

The following lemma concerning the switching signal σ generated by $\mathbb{S}$ can be proven as Lemma 1 in [14].

Lemma 2 *Let $\mathcal{T} := \{0 =: t_0, t_1, \ldots, t_j, \ldots\}$ be the sequence of switching times of σ. Then there exists a closed bounded subset $\mathcal{P}^* \subset \mathcal{P}$ containing p^* with the following properties.*

(i) For any $\varepsilon > 0$ there exist a finite switching time $t^* \in \mathcal{T}$ and a piecewise-constant signal $\sigma^* : [0, \infty) \to \mathcal{P}^*$, whose switching times are a subset of $\mathcal{T}$, such that $|\sigma(t) - \sigma^*(t)| \le \varepsilon$ for all $t \ge t^*$;

(ii) For each $p \in \mathcal{P}^*$, $e_p \in \mathcal{L}_2[0, \infty)$.

The lemma is instrumental in proving the following theorem, which is the main result of the paper.

Theorem 1 *Let $\tilde{\mathbb{P}}$ be the process (5), unknown member of the family of nominal plant models $\mathbb{N}_p$, with $p \in \mathcal{P}$, where $\mathcal{P}$ is a closed, bounded subset of a real, finite-dimensional, normed linear space. Suppose that Assumptions 1, 2 and 3 hold and that the function* $\mathrm{sat}(\cdot)$ *is continuously differentiable in a neighborhood of the origin. Consider the supervisory control system described by the equations (21), along with the state dependent dwell time switching logic $\mathbb{S}$, with $\tau_{\mathrm{D}}(\cdot)$ satisfying (20). Then, for each set of initial conditions $x_{\mathbb{P}}(0)$, $x(0)$, $W(0) > 0$, $\sigma(0)$, the response of the supervisory control system exists for all $t \ge 0$ and all the continuous states converge to zero as t goes to infinity.*

Proof: Let ε be as in Assumption 3, and fix t^* and σ^* according to point (i) in Lemma 2. As a consequence of Assumption 3 (cf. Fact 1), we are guaranteed that the response of the supervisory control system (21) and all the continuous states are bounded for all finite t. In particular, for $t \ge t^*$, we know from point (i) in Lemma 2 that the switching signal $\sigma(\cdot)$ generated by $\mathbb{S}$ satisfies $|\sigma(t) - \sigma^*(t)| \le \varepsilon$. Note that if t_i and t_{i+1} are two consecutive switching times of σ, with $t_i \ge t^*$, then both σ and σ^* are constant for all $t \in [t_i, t_{i+1})$.
If we consider the differential equation in (21) under the feedback interconnection $y = y_{\sigma^*} - e_{\sigma^*}$, namely

$$\dot{x} = \bar{A}_{\sigma^*} x + b\,\mathrm{sat}(\overline{\chi}_\sigma(x)) - de_{\sigma^*} \ , \tag{23}$$

by Assumption 3 we have that the state of the switching controller satisfies for all $t \in [t_i, t_{i+1})$,

$$\alpha(|x(t)|) \le \tilde{\theta}_1(\tilde{\theta}_2(|x(t_i)|)e^{-(t-t_i)}) + \int_{t_i}^{t} \bar{\gamma}|e_{\sigma^*}(\tau)|^2 d\tau. \tag{24}$$

Denote $c_{qp^*} := c_q - c_{p^*}$. For any $q \in \mathcal{P}^*$, we can write

$$\begin{aligned} e_q &= y_q - y = y_q - (y_{p^*} - e_{p^*}) \\ &= (c_q - c_{p^*})x - e_{p^*} = c_{qp^*}x - e_{p^*} \ . \end{aligned} \tag{25}$$

Fix a basis $\{c_{p_1p^*}, \ldots, c_{p_mp^*}\}$ of the row-vector space $\{c_{qp^*} : \ q \in \mathcal{P}^*\}$ and define (as in the proof of Lemma 1 in [14]) the matrix $C = [c'_{p_1p^*} \ \cdots \ c'_{p_mp^*}]'$. Then there exists a bounded function $s : \mathcal{P}^* \to \mathbb{R}^m$ such that $s(q)C = c_q - c_{p^*}$. Set $\bar{e} := Cx$. Since the i-entry of $\bar{e}$ is $e_{p_i} - e_{p^*}$ which (cf. (ii) of Lemma 2) is a signal in $\mathcal{L}_2[0, \infty)$, $\bar{e} \in \mathcal{L}_2[0, \infty)$ as well. Note also that, for any $q \in \mathcal{P}^*$, $|e_q|^2 = |s(q)\bar{e} - e_{p^*}|^2 \le 2|s(q)\bar{e}|^2 + 2|e_{p^*}|^2$. Then, for any switching time $t_i \ge t^*$,

$$\begin{aligned} &\int_{t_i}^{\infty} |e_{\sigma^*(\tau)}(\tau)|^2 d\tau = \sum_{\ell=i}^{\infty} \int_{t_\ell}^{t_{\ell+1}} |e_{\sigma^*(t_\ell)}(\tau)|^2 d\tau \\ &\le \sum_{\ell=i}^{\infty} \int_{t_\ell}^{t_{\ell+1}} 2|s(\sigma^*(t_\ell))\bar{e}(\tau)|^2 d\tau + \int_{t_i}^{\infty} 2|e_{p^*}(\tau)|^2 d\tau \\ &\le k \int_{t_i}^{\infty} |\bar{e}(\tau)|^2 d\tau + 2\int_{t_i}^{\infty} |e_{p^*}(\tau)|^2 d\tau < \infty \ , \end{aligned}$$

for some suitable constant $k \ge 0$. Hence, from t^* system (23) switches among integral input-to-state stable systems and is driven by an $\mathcal{L}_2$ signal. This yields the convergence to zero of the state $x(t)$, in view of the following result whose proof is omitted and can be found in [3], Theorem 4.

Lemma 3 *Consider system (23) and assume that on each switching interval $[t_i, t_{i+1})$, $t_i \ge t^*$, it satisfies inequality (24). Let σ be generated by the state dependent switching logic $\mathbb{S}$. Then, for each $x_0 \in \mathbb{R}^{2n_\nu}$, for each input $e_{\sigma^*}(\cdot) \in \mathcal{L}_2$, the solution $x(\cdot)$ of (23) starting from the initial condition $x(0) = x_0$ and under the input $e_{\sigma^*}(\cdot)$ is such that $\lim_{t \to \infty} |x(t)| = 0$.*

The convergence to zero of the remaining continuous states of the supervisory control system descends from the detectability of the plant using standard arguments (cf. [9], [7]). ■

Remark. From the proof, it is understood that on the interval $[0, t^*)$, the solution $x(\cdot)$ is guaranteed to

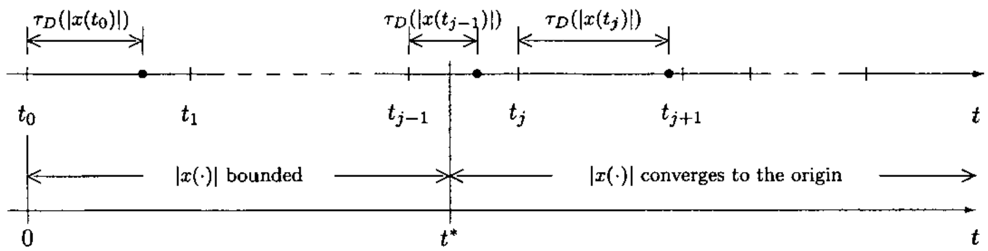

Figure 3: Timing diagram.

be bounded. It is starting from t^* that convergence to zero of $x(\cdot)$ is guaranteed as well (see Figure 3). $\triangleleft$

7 Conclusions

In this paper we have proposed a solution to the problem of supervisory control of largely uncertain systems under input constraints, in the case in which the unknown process belongs to a continuum of nominal models. The analysis rests on the concept of robust integral input-to-state stability. Our design achieves global regulation of the state to zero for plants which are open-loop unstable but not exponentially unstable.

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